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An objective, local, hidden-variables theory of the Clauser–Horne experiment

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Abstract. An objective, local, hidden-variables theory of the Clauser–Horne photon correlation experiment is derived using the method previously used to describe the Bohm–EPR thought experiment.

We have shown elsewhere (Liddy 1983) that it is possible to construct an objective, local, hidden-variables theory (OLHV, see Clauser and Horne (1975) for a definition of these terms) that reproduces the quantum mechanical predictions for the Bohm–EPR thought experiment (Bohm 1951). However, almost all the experiments that have been performed to test these predictions have employed a variant of the thought experiment, produced by Clauser and Horne (1975), which uses photons instead of spin-$\frac{1}{2}$ particles. (For a review see Clauser and Shimony 1978.) It therefore seems appropriate to look for an OLHV which describes this modified experiment.

The experiment essentially entails the production of two photons which travel in opposite directions and whose polarisations are strictly correlated. That is, if one of the photons were to pass through a polariser aligned in a particular direction then the second photon would pass through a similarly aligned polariser without fail. The photons then enter polarisers, whose settings are known, and we record whether or not the photons pass through the polarisers. We will use an idealised version of the experiment in which it is known that the photons have entered the polarisers and further that the polarisers are 100% efficient.

We will use the same method for constructing the OLHV as was used in Liddy (1983). To be able to do this, we will denote the two possible outcomes of any measurement by numbers which have the same absolute value. Thus we will say that the outcome of the event 'the photon passes the polariser' is 1 and that the outcome is -1 otherwise. The quantum mechanical expectation for the outcome of a measurement on photon A, by a polariser aligned in direction $\boldsymbol{a}$ (all vectors used will be of unit length), is thus (see Clauser and Shimony 1978, pp 1906–7)

$$E[A_a]=0, \tag{1}$$

while the conditional expectation for a second measurement in direction $\boldsymbol{b}$, given that the photon passed the first polariser, is

$$E[A_b|A_a=1]=2(\boldsymbol{a}\cdot\boldsymbol{b})^2-1. \tag{2}$$

The expectation given by quantum mechanics for the product of the outcomes of measurements on each of the correlated particles is

$$E[A_aB_b]=2(\boldsymbol{a}\cdot\boldsymbol{b})^2-1. \tag{3}$$

These three results must be reproduced by any OLHV of the experiment under consideration. Before constructing our OLHV it must be stated that no attempt will be made at explaining or searching for the mechanisms involved. The sole aim of this exercise is to show that the predictions of quantum mechanics can be reproduced by OLHVS.

Let us now suppose that all photons have a definite polarisation, denoted by the unit vector $\boldsymbol{P}$. As light is a transverse wave this vector must always be perpendicular to the direction of motion. Further suppose that a photon passing through a polariser will have its polarisation changed so that

$$\boldsymbol{P}\to\boldsymbol{a}, \tag{4}$$

where $\boldsymbol{a}$ is the orientation of the polariser. Using this and (2) we can immediately identify the expectation of a measurement on a single photon as

$$E[A_a|\boldsymbol{P}]=2(\boldsymbol{a}\cdot\boldsymbol{P})^2-1. \tag{5}$$

The conditional variance is then given by

$$V[A_a|\boldsymbol{P}]=E[A_a^2|\boldsymbol{P}]-(E[A_a|\boldsymbol{P}])^2=4(\boldsymbol{a}\cdot\boldsymbol{P})^2|\boldsymbol{a}\times\boldsymbol{P}|^2. \tag{6}$$

Also if the photons take all values of $\boldsymbol{P}$ with equal probability (i.e. the light source is unpolarised) we find that

$$E[A_a]=\int E[A_a|\boldsymbol{P}]\,\mathrm{d}\boldsymbol{P}=\frac{1}{2\pi}\int_{-\pi}^{\pi}\cos 2\phi\,\mathrm{d}\phi=0, \tag{7}$$

where the coordinates have been chosen so that the photon moves in the y direction, that $\boldsymbol{a}=\boldsymbol{k}$, and that $\boldsymbol{P}=\sin\phi\boldsymbol{i}+\cos\phi\boldsymbol{k}$ ($\boldsymbol{i},\boldsymbol{j},\boldsymbol{k}$ are the usual Cartesian unit vectors). Thus our choice of conditional expectation, in the case of a single photon, satisfies both (1) and (2).

Now consider the Clauser–Horne experiment. We have two photons, A and B, which are strictly correlated. That is, the outcomes of measurements by parallel or perpendicularly oriented polarisers are related by

$$B_a=A_a=-A_{\perp a}, \tag{8}$$

where $\perp a$ denotes those two vectors which are perpendicular to $\boldsymbol{a}$ and normal to the direction of motion. Thus we have

$$E[B_a|\boldsymbol{P}]=E[A_a|\boldsymbol{P}]=-E[A_{\perp a}|\boldsymbol{P}]. \tag{9}$$

By requiring that the polarisation vector for photon B be the same as that for photon A we can ensure that equations (9) are satisfied.

We may now write the conditional joint expectation as

$$E[A_aB_b|\boldsymbol{P}]=E[A_a|\boldsymbol{P}]E[B_b|\boldsymbol{P}]+\mathrm{Cov}[A_a,B_b|\boldsymbol{P}], \tag{10}$$

where $\mathrm{Cov}[A_a,B_b|\boldsymbol{P}]$ is, by definition, the conditional covariance. Note that

$$\mathrm{Cov}[A_a,B_a|\boldsymbol{P}]=V[A_a|\boldsymbol{P}]. \tag{11}$$

The conditional covariance is constrained by the inequality (see e.g. Rao 1965)

$$|\mathrm{Cov}[A_a,B_b|\boldsymbol{P}]|\leqslant(V[A_a|\boldsymbol{P}]V[B_b|\boldsymbol{P}])^{1/2}, \tag{12}$$

and must be such as to enable the averaged conditional joint expectation to satisfy (3). In order to show that there exists a conditional covariance which satisfies these conditions we will write it as

$$\mathrm{Cov}[A_a, B_b|\boldsymbol{P}] = \rho(\theta)(V[A_a|\boldsymbol{P}]V[B_b|\boldsymbol{P}])^{1/2}, \tag{13}$$

where θ is the small angle between $\boldsymbol{a}$ and $\boldsymbol{b}$ and $\rho(\theta)$ is, by definition, the correlation coefficient. Our choice of the dependence of ρ on θ alone enables us to determine it uniquely, as well as to satisfy (3) and (12). Since our aim is to show the possibility of an OLHV, this restriction does not weaken the argument and is justified.

Using (10) and (13), and assuming (3), we find that

$$\rho(\theta) = (\pi \cos 2\theta)/[2 \sin 2\theta + (\pi - 4\theta) \cos 2\theta], \qquad 0 \leqslant \theta \leqslant \pi/2, \tag{14}$$

where we have again assumed that the light source is unpolarised. As $|\rho(\theta)| \leqslant 1$, (12) is satisfied, thus showing that our OLHV is viable. The conditional joint expectation is then

$$E[A_aB_b|\boldsymbol{P}] = [2(\boldsymbol{a}\cdot\boldsymbol{P})^2 - 1][2(\boldsymbol{b}\cdot\boldsymbol{P})^2 - 1] + \pi|\boldsymbol{a}\cdot\boldsymbol{P}||\boldsymbol{a}\times\boldsymbol{P}||\boldsymbol{b}\cdot\boldsymbol{P}||\boldsymbol{b}\times\boldsymbol{P}|[2(\boldsymbol{a}\cdot\boldsymbol{b})^2 - 1]$$
$$\times\{|\boldsymbol{a}\cdot\boldsymbol{b}||\boldsymbol{a}\times\boldsymbol{b}| + [\pi/4 - \cos^{-1}(\boldsymbol{a}\cdot\boldsymbol{b})][2(\boldsymbol{a}\cdot\boldsymbol{b})^2 - 1]\}^{-1}. \tag{15}$$

We may now calculate the joint probabilities. Firstly the quantum mechanical prediction for the probability of a photon passing a polariser aligned in direction $\boldsymbol{b}$ given that it has passed a polariser aligned in direction $\boldsymbol{a}$ is

$$p[A_b = 1|A_a = 1] = (\boldsymbol{a}\cdot\boldsymbol{b})^2. \tag{16}$$

As in (5), we can therefore identify the probability that a photon polarised in direction $\boldsymbol{P}$ passes an analyser set for direction $\boldsymbol{a}$ is

$$p[A_a = 1|\boldsymbol{P}] = (\boldsymbol{a}\cdot\boldsymbol{P})^2. \tag{17}$$

When averaged over all values of $\boldsymbol{P}$ this gives

$$p[A_a = 1] = \tfrac{1}{2}\pi \int_{-\pi}^{\pi} \cos^2\phi \, \mathrm{d}\phi = \tfrac{1}{2}, \tag{18}$$

again giving agreement with quantum mechanics.

The joint probabilities for the Clauser–Horne experiment can now be obtained. It follows from (5), (10) and (17) that

$$p[A_a = \alpha \cap B_b = \beta|\boldsymbol{P}] = p[A_a = \alpha|\boldsymbol{P}]p[B_b = \beta|\boldsymbol{P}] + \tfrac{1}{4}\alpha\beta\, \mathrm{Cov}[A_a, B_b|\boldsymbol{P}], \tag{19}$$

where $\alpha, \beta \in \{-1, 1\}$. Averaging over $\boldsymbol{P}$ again returns the quantum mechanical result. The joint probabilities also satisfy the inequalities

$$0 \leqslant p[A_a = \alpha \cap B_b = \beta|\boldsymbol{P}] \leqslant p[A_a = \alpha|\boldsymbol{P}], p[B_b = \beta|\boldsymbol{P}]. \tag{20}$$

Formally, we can say that we have determined a measure space $M = (X, \mathscr{A}, m)$ for our hidden-variables system where X is the product space of the event spaces A_a, B_b and S_1 (i.e. the vectors $\boldsymbol{P}$ are represented as points on the surface of a unit circle), $\mathscr{A}$ is the product of the power set of $A_a \times B_b$ with the Borel σ-field of S_1, and the measure m is defined by

$$m(X_i) = m(A_{ai} \times B_{bi} \times \mathrm{d}\Omega_i) = p[A_a = \alpha_i \cap B_b = \beta_i|\boldsymbol{P}_i](\mathrm{d}\theta_i)/2\pi, \tag{21}$$

where $\mathrm{d}\Omega_i$ is the infinitesimal solid angle within which $\boldsymbol{P}_i$ lies. This satisfies all the requirements of a probability space and returns the quantum mechanical probabilities as marginals.

This probability space is not unique, nor do we have a physical justification for it. We have merely tried to show that the Clauser–Horne experiment may be described by an OLHV and that these experiments cannot differentiate quantum mechanics from OLHV.

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